XLIII. *A Letter to* James Weſt, *Eſq; Preſident of the Royal Society, containing the Inveſtigations of Twenty Caſes of Compound Intereſt, by* J. Robertſon, *Lib. R. S.*

S I R,

Read Dec. 13, 1770.

IT is well known that the conſideration of Compound Intereſt is of great utility in the Computations reſpecting the values of Penſions, Annuities, Reverſions, and other affairs relating to Money concerns; and therefore, many Mathematicians have beſtowed ſome time on this uſeful ſubject, and have endeavoured to diſcover practical methods for ſolving the various caſes that might occur: And in theſe inquiries, the uſe of Logarithms has been found of ſingular ſervice in facilitating the operations.

The late William Jones, Eſq; F. R. S. among the variety of Mathematical matters to which he gave attention, conſidered the buſineſs of Compound Intereſt fully, and did, many years ago, cauſe to be engraved on a Copper-plate, more caſes in Intereſt than had been exhibited before that time: Several copies of impreſſions from that plate were diſtributed among his friends; to whom it appeared that he had treated this ſubject in a more extenſive manner than had been done by other Mathematicians.

The

The Theorems, or Rules, for the Cafes of Compound Intereft, without their inveftigations, were inferted by Mr. Jones, in the quarto edition of Logarithms, publifhed by Gardiner; and the Rules were alfo communicated to Mr. Dodfon, who publifhed them, by Mr. Jones's leave, with examples to illuftrate the ufe of his Antilogarithmic Tables: But the inveftigations of thefe Theorems not having yet been made public, it is apprehended that Gentlemen curious in thefe fpeculations would be pleafed to fee them: And fhould the following Effay be approved of by you, who have been for many years acquainted with the Finances of this Nation, and the importance of a ready knowledge in computing the Intereft on the National Loans, you will be pleafed to communicate it to the Royal Society.

I am, SIR,

Your moft humble fervant,

John Robertfon.

In this fubject five particulars are taken into confideration.

1ft, The Annuity, Rent, or Penfion.

2d, The Times that Annuity, Rent, or Penfion is to continue.

3d, The Rate of Intereft ufed in the computation.

4th, The Amount of thofe Rents, and their Intereft, when they are forborn to be received any times after they are due.

5th, The prefent worth of thofe Rents, fome times before they are due; or, of a Sum to be received before it is due, Difcount being allowed.

And

And the inveſtigations naturally fall under two heads.

Firſt, The Conſideration of Amounts.

Secondly, The Conſideration of Diſcounts.

Under the firſt head an Equation is to be obtained between the Annuity, Time, Rate and Amount, from the known proportion that ſubſiſts between ſums of money put to intereſt, during the ſame length of time, and the amounts of the principal and intereſt together.

Under the ſecond head another Equation is to be formed between the Annuity, Rate, Time and preſent Worth, from the known proportion that ſubſiſts between the ſums diſcounted, and their preſent worths, when done for the ſame time.

As theſe Equations involve quantities common to both of them, therefore other Equations may be thence deduced, containing all the five terms before ſpecified. And hence, any three of the five terms being given, the other two are to be found, which admits of 20 Caſes.

Some of theſe Caſes will produce Adfected Equations, where the index of the higheſt power of the unknown quantity will be the number of times the Rent is to continue, or to be paid: Therefore, the ſolution of thoſe Caſes will be given by a method of Approximation, as no better way has yet been diſcovered for the ſolution of Adfected Equations, in numbers, above the third or fourth degree.

In the following inveſtigations,

Let $a =$ Annuity, Rent, or Penſion.

$n =$ Number of times that intereſt is to be paid for the annuity, or ſum lent.

$r =$ Rate of intereſt of 1 £. for 1 time.

$m =$ Amount of the annuity, or ſum lent for n-times, at r intereſt.

$p =$ Principal ſum uſed, or preſent worth of a ſum before it is due.

Of Compound Intereſt.

Firſt, In Amounts,

Let $q = 1 + r =$ Amount of 1 £. for 1 time.

Now, $a =$ laſt year's amount.

And $1 : q :: a : aq =$ laſt but one year's amount.

$1 : q :: aq : aq^2 =$ laſt but two year's amount.

$1 : q :: aq^2 : aq^3 =$ laſt but 3.

And ſo on to $aq^{n-1} =$ firſt year's amount.

Therefore $a + aq + aq^2 + aq^3$ &c. $+ aq^{n-1} = m$.

But $a : aq :: m - aq^{n-1} : m - a$. Euc. 12. v.

Then $m - a = mq - aq^n$

Or $m - a = m + mr - aq^n$

Therefore $aq^n - a = mr$ Put $A = q^n$.

Then $\overline{A - 1} \times a = mr$

Therefore $A = \frac{mr + a}{a}$

Hence $a = \frac{mr}{A - 1}$

$$m = \frac{\overline{A - 1} \times a}{r}$$

$$r = \frac{\overline{A - 1} \times a}{m}.$$

Second,

Second, In Discounts.

Since $q : 1 :: a : \frac{a}{q}$ = 1st year's present worth.

$q : 1 :: \frac{a}{q} : \frac{a}{q^2}$ = 2d year's present worth.

$q : 1 :: \frac{a}{q^2} : \frac{a}{q^3}$ = 3d.

And so on to $\frac{a}{q^n}$ = nth years present worth.

Therefore $\frac{a}{q} + \frac{a}{q^2} + \frac{a}{q^3} +$, &c. to $\frac{a}{q^n} = p$.

But $\frac{a}{q} : \frac{a}{q^2} :: p - \frac{a}{q^n} : p - \frac{a}{q}$. Euc. 12. V.

Therefore $\frac{pq^n - a}{q^n} = pq - a$.

Or $\frac{pq^n - a}{q^n} = p + pr - a$.

Therefore $aq^n - a = prq^n$.

Or $\overline{A - 1} \times a = prA$.

Therefore $A = \frac{a}{a - rp}$

Hence $\overline{A - 1} \times a = mr = prA$.

Therefore $A = \frac{mr + a}{a} = \frac{a}{a - rp} = \frac{m}{p} = \overline{1 + r}^n$.

Case I. Given a, m, p: Required r.

Since $A = \frac{m}{p}$.

Therefore $r = \frac{\overline{A - 1} \times a}{m}$.

Case II.

Case II. Given p, m, n: Required r.

Since $\overline{r+1}|^{n} = \frac{m}{p}$.

Therefore $r = \overline{\frac{m}{p}}|^{\frac{1}{n}} - 1$.

Case III. Given a, m, n: Required r.

Since $\overline{A-1} \times a = mr$.

Therefore $\left(\frac{A-1}{nr} =\right) \frac{m}{na} = \frac{\overline{1+r}|^{n} - 1}{nr}$.

Now $\overline{1+r}|^{n} = 1 + nr + n \times \frac{n-1}{2} r^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} r^3$, &c.

Therefore $\frac{m}{na} = 1 + \frac{n-1}{2} r + \frac{n-1}{2} \times \frac{n-2}{3} r^2$, &c.

And $\overline{\frac{m}{an}}|^{\frac{2}{n-1}} = \overline{1 + \frac{n-1}{2} r + \frac{n-1}{2} \times \frac{n-2}{3} r^2}|^{\frac{2}{n-1}}$;
which, by the Binomial Theorem, will become
$= 1 + r + \frac{n+1}{12} r^2$ (nearly).

Let $D = \left(\overline{\frac{m}{na}}|^{\frac{2}{n-1}} =\right) 1 + r + \frac{n+1}{12} r^2$,

Then $r^2 + \frac{12}{n+1} r = \overline{D-1} \times \frac{12}{n+1}$, Let $2E = \frac{12}{n+1}$,

Then $r + E = \left(\sqrt{2 \times \overline{D-1} + E \times E} =\right) F$.

Therefore $r = F - E$.

In this Solution, 1st, find $D = \overline{\frac{m}{na}}|^{\frac{2}{n-1}}$,

2d, find $E = \frac{6}{n+1}$,

3d, find $F = \sqrt{2 \times \overline{D-1} + E \times E}$,

4th, find $r = F - E$.

 Case IV.

Case IV. Given a, p, n; Required r.

Since $\overline{A-1} \times a = Apr$.

Therefore $\frac{p}{na} = \frac{A-1}{Anr} = \frac{1-A^{-1}}{nr} = \frac{1-\overline{1+r}|^{-n}}{nr}$.

Now, $\overline{1+r}|^{-n} = 1 - nr + n \times \frac{n+1}{2} r^2 - n \times \frac{n+1}{2} \times \frac{n+2}{3} r^3 +$ &c,

Therefore $\frac{p}{na} = 1 - \frac{n+1}{2} r + \frac{n+1}{2} \times \frac{n+2}{3} r^2$, nearly.

Then $\left.\frac{p}{na}\right|^{-\frac{2}{n+1}} = \overline{1 - \frac{n+1}{2} r + \frac{n+1}{2} \times \frac{n+2}{3} r^2}|^{-\frac{2}{n+1}}$;

which, by the Binomial Theorem, will become

$$= 1 + r - \frac{n-1}{12} rr, \text{ nearly.}$$

Now, $\left.\frac{p}{na}\right|^{-\frac{2}{n+1}} = \left.\frac{na}{p}\right|^{\frac{2}{n+1}}$.

Let $G = \left(\left.\frac{na}{p}\right|^{\frac{2}{n+1}} =\right) 1 + r - \frac{n-1}{12} r^2$.

Therefore $rr - \frac{12}{n-1} r = -\overline{G-1} \times \frac{12}{n-1}$.

Let $2H = \frac{12}{n-1}$.

Then $r - H = \left(\sqrt{HH - 2 . \overline{G-1} \times H} =\right) K$.

Therefore $r = H - K$.

Case V. Given a, n, r; Required m.

Since $\overline{A-1} \times a = mr$. Therefore $m = \frac{\overline{A-1} \times a}{r}$.

Caſe VI. Given p, n, r; Required m.

Since $A = \frac{m}{p}$. Therefore $m = pA$.

Caſe VII. Given a, p, r; Required m.

Since $\frac{m}{p} = \frac{a}{a - rp}$. Therefore $m = \frac{ap}{a - rp}$.

Caſe VIII. Given a, p, n; Required m.

Find $G = \overline{\frac{na}{p}}\big|^{\frac{2}{n+1}}$; $H = \frac{3}{\frac{1}{2}n - 1}$; $K = \sqrt{H - 2\overline{G - 1} \times H}$.

Now, $H - K = r$ by 4th. But $\overline{r + 1}|^{n} = \frac{m}{p}$.

Therefore $m = \overline{r + 1}|^{n} \times p$.

Caſe IX. Given a, n, r; Required p.

Since $\overline{A - 1} \times a = prA$. Therefore $p = \frac{\overline{A - 1} \times a}{Ar}$.

Caſe X. Given m, n, r; Required p.

Since $A = \frac{m}{p}$. Therefore $p = \frac{m}{A}$.

Caſe XI. Given a, m, r; Required p.

Since $\frac{mr + a}{a} = \frac{m}{p}$. Therefore $p = \frac{ma}{mr + a}$.

Cafe XII. Given a, m, n; Required p.

Find $D=\frac{\overline{m}|^{\frac{2}{n-1}}}{na}$; $E=\frac{3}{\frac{1}{2}n+1}$; $F=\sqrt{\overline{2D-1}+E\times E}$

Now, $F-E=r$, by 3d. But $\overline{1+r}|^{n}=\frac{m}{p}$.

Therefore $p=\frac{m}{\overline{1+r}|^{n}}$.

Cafe XIII. Given p, n, r; Required a.

Since $\overline{A-1}\times a=prA$. Therefore $a=\frac{prA}{A-1}$.

Cafe XIV. Given p, m, n; Required a.

Since $A=\overline{r+1}|^{n}=\frac{m}{p}$. Therefore $r+1=\overline{\frac{m}{p}}|^{\frac{1}{n}}$.

Hence $A-1$, and $B-1$ $(=r)$ are known.

Therefore $a=\frac{m\times\overline{B-1}}{A-1}$.

Cafe XV. Given m, n, r; Required a.

Since $\overline{A-1}\times a=mr$. Therefore $a=\frac{mr}{A-1}$.

Cafe XVI. Given p, m, r; Required a.

Since $A=\frac{m}{p}$, $A-1$ is given.

Therefore $a=\frac{mr}{A-1}$.

Cafe XVII.

Cafe XVII. Given m, p, r; Required n.

Put L, for logarithm; and $\grave{\text{L}}$, for arith. comp. of a log.

Since $\overline{1+r}|^n = \frac{m}{p}$. Therefore $n = \frac{\text{L}, m + \grave{\text{L}}, p}{\text{L}, \overline{r+1}}$.

Cafe XVIII. Given a, p, r; Required n.

Since $\overline{1+r}|^n = \frac{a}{a - pr}$. Therefore $n = \frac{\text{L}, a - \text{L}, \overline{a - pr}}{\text{L}, \overline{r+1}}$.

Cafe XIX. Given a, p, m; Required n.

Since $\text{A} = \left(\overline{r+1}|^n = \right) \frac{m}{p}$,

Then $(\text{L}, \overline{r+1}|^n =)\ \text{L}, m - \text{L}, p = \text{L}, \text{A}$;

Hence A, and A — 1, are known.

Alfo $\text{L}, \overline{\text{A} - 1} + \text{L}, a + \grave{\text{L}}, m = \text{L}, \overline{\text{B} - 1}$ (by Cafe 14th):

Hence $r = (\text{B} - 1)$, and $r + 1$, are known.

Therefore $n = \frac{\text{L}, \text{A}}{\text{L}, \overline{r+1}}$.

Cafe XX. Given a, m, r; Required n.

Since $(\text{A} =) \frac{mr + a}{a} = \overline{1+r}|^n$.

Therefore $n = \frac{\text{L}, \overline{mr + a} - \text{L}, a}{\text{L}, \overline{1+r}}$.